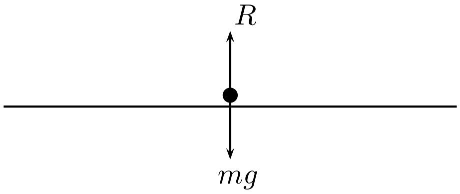
0J2 - Mechanics Lecture Notes 3

Normal Reaction

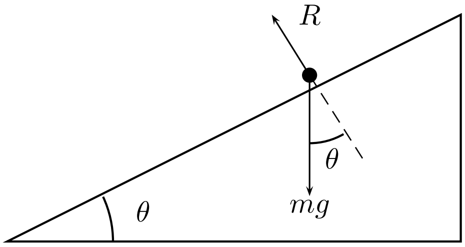
a) Horizontal surface

When a mass m is moving, or stationary, on a horizontal surface there is no motion in the vertical direction. Since there is no vertical acceleration there is no net force in the vertical direction (Newton's 2nd law). Thus the force of gravity (weight) must be balanced by an equal and opposite force exerted by the surface on the particle. This is called the *Normal Reaction* R. Clearly the direction of this force is normal to the surface.



b) Inclined surface

If it is moving (or stationary) on a surface inclined at an angle θ to the horizontal, then again the surface exerts a force R which is normal (i.e. perpendicular) to the surface, the *Normal Reaction*.

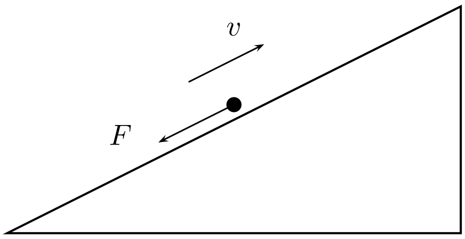


Since there is no motion in the direction normal to the surface, the total force in this direction must be zero. This means that the component of the weight in this direction must be equal and opposite to R so in this case

$$R = mg\cos\theta$$

Friction in Dynamics

If a body is moving along a surface (horizontal or inclined) which is not smooth then there is a friction force F. This always acts in the opposite direction to the velocity.



The magnitude of the friction force F is proportional to the magnitude of the normal reaction R. We use the Greek symbol μ for the constant of proportionality so

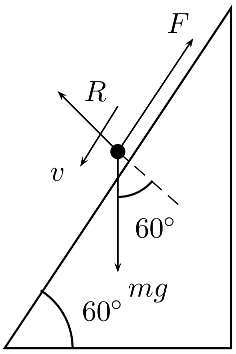
$$F = \mu R$$

and μ is called the coefficient of (dynamic) friction.

F is always independent of the velocity v.

There is also a coefficient of static friction if there is no motion. This is slightly different – see later.

Example A particle of mass 3 kg is sliding down a surface inclined at an angle 60° to the horizontal. The coefficient of friction $\mu = 0.3$. What is the friction force?
(Take $g = 9.81 \text{ ms}^{-2}$.)



Normal to the surface there is no motion so the total force must be zero in this direction

$$R - mg\cos 60^\circ = 0$$

$$\therefore \quad R = mg\cos 60^\circ = 3 \times 9.81 \times \frac{1}{2} \approx 14.72 \text{ N}$$

The friction force is

$$F \;=\; \mu R \;\approx\; 0.3 \times 14.72 \;=\; 4.415 \text{ N}$$

and this acts parallel to the plane in an upwards direction.

Impulses, Momentum and Collisions

If a force acts on a particle from time t_1 to time t_2 we say that the particle receives an impulse I defined by

$$\boxed{I \;=\; \int_{t_1}^{t_2} F\,dt} \qquad (1)$$

The units are force × time i.e. N s.

Example If a force $F \;=\; (4+2t)$ N acts from $t=1$ to $t=3$ find the impulse.

Here the force is varying in time and the formula gives

$$I \;=\; \int_1^3 (4+2t)\,dt \;=\; [4t+t^2]_1^3 \;=\; (12+9)-(4+1) \;=\; 16 \text{ N s}$$

Special case – constant force

Often the force is constant so

$$I \;=\; \int_{t_1}^{t_2} F\,dt \;=\; F\int_{t_1}^{t_2} dt \;=\; F(t_2-t_1) \;=\; FT \qquad (2)$$

where $T \;=\; t_2 - t_1$ is the total time.

End of special case.

Now, by Newton's 2nd law $F \;=\; ma \;=\; m\dfrac{dv}{dt}$.

If the force is not constant but the mass m is, then integrate both sides with respect to t

$$\int_{t_1}^{t_2} F\,dt \;=\; m\int_{t_1}^{t_2} \frac{dv}{dt}\,dt$$

$$I \;=\; m[v]_{t_1}^{t_2} \;=\; mv_2 - mv_1$$

where v_1 is the velocity at t_1 and v_2 is the velocity at t_2 .

Momentum

Momentum is defined as mass × velocity

$$\boxed{M \;=\; mv}$$

so for the constant mass case above we have

$$I \;=\; mv_2 - mv_1 \quad \Rightarrow \quad \text{Impulse} \;=\; \text{change in momentum.}$$

Impulses are frequently used where we have a large force acting for a small time. Examples are 1. the blow of a hammer on a nail, 2. Collision of two snooker balls.

Example A hammer of mass 0.8 kg is moving at 12 ms^{-1} when it strikes a nail and comes to rest. What is the impulse on the hammer? If the impulse lasts for 0.05 seconds, what is the average force of the hammer on the nail?

$$I \;=\; \text{change in momentum} \;=\; 0 - 0.8 \times 12 \;=\; -\,9.6 \text{ Ns}$$

Let the average force on the <u>hammer</u> be f, so using impulse = force × time for a constant force we have

$$\begin{aligned} f \times 0.05 &= -\,9.6 \\ f &= -\,192 \text{ N} \end{aligned}$$

The force on the <u>nail</u> is equal and opposite to that on the hammer (N3) so the average force on the nail is 192 N.

<u>Note</u> Since the force of body A on body B is equal and opposite to the force of B on A, and the time is the same for both, the <u>impulses are equal and opposite</u>.

Collisions of particles

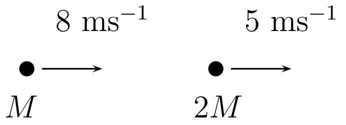
When two particles collide the impulses are equal and opposite so the change in momentum of one is equal and opposite to the change in momentum of the other. Thus there is no change in the total momentum, and we have a conservation law:-

<u>Conservation of linear momentum</u>

'In a given direction, if no external forces act on a system then the total momentum of the system, in that direction, remains unchanged.'

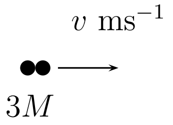
Example 1 A railway truck of mass M and velocity 8 ms^{-1} moves in a straight line to a truck of mass $2M$ moving at 5 ms^{-1} in the same direction. On impact they couple together. Find the final velocity v.

Before



Momentum $= 8M + 5(2M) = 18M.$

After

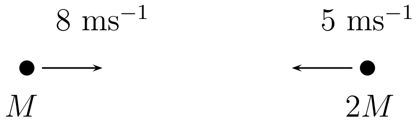


Momentum $= v(3M).$

By conservation of momentum $3Mv = 18M. \quad \therefore \quad v = 6 \ \mathrm{ms}^{-1}.$

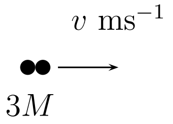
Example 2 What happens if they are initially moving in opposite directions?

Before



Momentum $= 8M - 5(2M) = -2M.$

After



By conservation of momentum $3Mv = -2M. \quad \therefore \quad v = -2/3 \ \mathrm{ms}^{-1}.$

Now consider the KE in Ex 1.

$$\begin{aligned} \text{Before:} \quad \text{KE} &= \frac{1}{2}M8^2 + \frac{1}{2}(2M)5^2 \quad = 57M \quad \text{joules} \qquad (1) \\ \text{After:} \quad \text{KE} &= \frac{1}{2}(3M)6^2 \quad = 54M \quad \text{joules} \qquad (2) \end{aligned}$$

Clearly (1) is not the same as (2).
The PE is the same before and after.

Therefore the total energy is NOT conserved in collisions, or in general when we have impulses. (Although momentum IS conserved).

– some energy is lost in sound – the 'bang' of the collision and some is converted to heat.

The impulse during the collision is the change in momentum of one of the bodies. The other one receives an equal and opposite impulse.

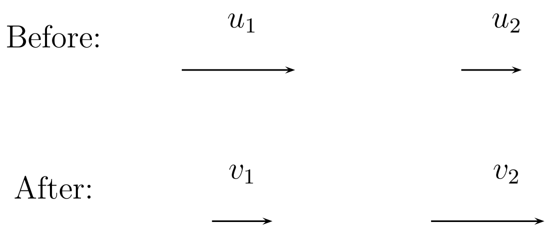
In Ex 1, considering only the mass M,
Momentum before $= 8M$
Momentum after $= 6M$
so the impulse on mass M is $-2M$ Newton seconds.

Law of Restitution

In the two previous examples we assumed that the particles joined together at the collision and then moved with the same velocity.

In general this does not happen and after the collision the particles will have different velocities.

Suppose we have two bodies moving with velocities u_1 and u_2 before the collision.
Here we measure velocities in the same direction (to the right, say). If the bodies are moving in opposite directions then u_2 will be negative. Let the two velocities after the collision be v_1 and v_2, again measured to the right.



Experimentally, we find that for any two colliding bodies the 'separation speed' is proportional to the 'approach speed'.

$$\begin{aligned} \text{approach speed} &= \text{difference in velocity before the collision} = u_2 - u_1 \\ \text{separation speed} &= \text{difference in velocity after the collision} = v_2 - v_1 \end{aligned}$$

The constant of proportionality has the symbol e and is called the coefficient of restitution.

Since the two bodies are moving towards each other before and away from each other after we write

$$(v_2 - v_1) = -e\,(u_2 - u_1)$$

Notes

1. All velocities measured in the same direction.
2. e is a positive number and $0 \leq e \leq 1$.

If $e = 1$ then we say the collision is 'perfectly elastic'. It can be shown that in this case energy IS conserved.

If $e = 0$ then then we say the collision is 'perfectly inelastic'. This is the case where $v_2 = v_1$ and the particles 'coalesce', as in Ex 1.

Proof that if $e = 1$ then energy is conserved in the collision.

If $e = 1$ then $v_2 - v_1 = -(u_2 - u_1)$ so $v_2 + u_2 = v_1 + u_1$ (1)

By conservation of momentum

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ \therefore \quad m_2(u_2 - v_2) &= m_1(v_1 - u_1) \qquad (2) \end{aligned}$$

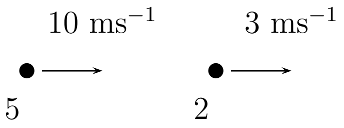
Multiply (2) by (1)

$$\begin{aligned} m_2(u_2^2 - v_2^2) &= m_1(v_1^2 - u_1^2) \\ \therefore \quad m_1u_1^2 + m_2u_2^2 &= m_1v_1^2 + m_2v_2^2 \\ \therefore \quad \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2. \end{aligned}$$

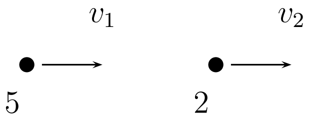
Thus the KE is the same before and after.
Conclusion: For perfectly elastic ($e = 1$) collisions energy is conserved as well as momentum. For other collisions ($e < 1$) only momentum is conserved.

Example A particle of mass 5 kg travelling at 10 ms^{-1} strikes a particle of mass 2 kg travelling at 3 ms^{-1} in the same direction. If the coefficient of restitution $e = 0.5$, find the velocities after impact.

Before



After



Here $m_1 = 5$, $m_2 = 2$, $u_1 = 10$, $u_2 = 3$.

Using the law of restitution

$$v_2 - v_1 = -e(u_2 - u_1) = -0.5 \times (-7) = 3.5 \qquad (1)$$

Using conservation of momentum

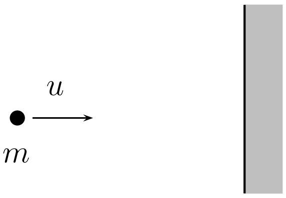
$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ 5 \times 10 + 2 \times 3 &= 5v_1 + 2v_2 \\ 5v_1 + 2v_2 &= 56 \qquad (2) \end{aligned}$$

From (1) $v_2 = v_1 + 3.5$ so

$$\begin{aligned} 5v_1 + 2(v_1 + 3.5) &= 56 \\ 7v_1 &= 49 \\ \therefore \quad v_1 &= 7 \text{ ms}^{-1}. \\ \text{and so} \quad v_2 &= 10.5 \text{ ms}^{-1}. \end{aligned}$$

Notice that m_1 slows down and m_2 speeds up.

Impacts with fixed surfaces



If a particle of mass m strikes a fixed surface, then we cannot use conservation of momentum, since we cannot talk about the 'momentum' of a surface!

In the diagram the surface is a wall, and we do know that the wall does not move.

In fact the impulse at the collision causes the particle to bounce off the wall, and the law of restitution applies.

Let u be the velocity of approach to the wall (left to right) and v be the velocity away from the wall after impact. N.B. Here v is the velocity from right to left!

Thus we have

$$\begin{aligned} u_1 &= u, \quad u_2 = 0 \\ \text{and} \quad v_1 &= -v, \quad v_2 = 0 \end{aligned}$$

u_2 and v_2 being zero since the wall does not move.

Using the law of restitution

$$\begin{aligned} v_2 - v_1 &= -e\,(u_2 - u_1) \\ 0 + v &= -e\,(0 - u) \end{aligned}$$

$$\therefore \quad \boxed{\text{v} = \text{eu}}$$

Example A particle of mass 3 kg travelling at 4 ms^{-1} strikes a wall. If the coefficient of restitution is $e = 0.2$ find the velocity after the impact and the energy lost in the collision.

Before: $u = 4$ (towards the wall)
After: $v = +e\,u = 0.8$ ms^{-1} (away from the wall)

The energy (KE) before $= \frac{1}{2}3 \times 4^2 = 24$ joules.
The energy after $= \frac{1}{2}3 \times (0.8)^2 = 0.96$ joules.

So the energy lost is 23.04 joules.

N.B. The velocity before impact means the velocity immediately before impact. Likewise the velocity after impact means the velocity immediately after impact. If the velocity of the particle is changing, i.e. it is accelerating or decelerating we need the velocity at impact.

Example A ball of mass m is dropped from rest at a height h above the (horizontal) ground. The coefficient of restitution is $e = 1/3$. Find the height of the first bounce.

There are three stages
1. Descent to ground.
By conservation of energy it hits the ground with a velocity u where

$$\tfrac{1}{2}mu^2 = mgh \quad \text{so} \quad u = \sqrt{2gh} \text{ (downwards).}$$

2. Collision.
Using the law of restitution the velocity after impact is $v = e\,u = (1/3)\sqrt{2gh}$ (upwards).

3. Up to top of first bounce.
It now rises to a height h_2 where, by conservation of energy,

$$mgh_2 = \frac{1}{2}mv^2 = \frac{1}{2}m\,\frac{1}{9}(2gh)$$

Thus $h_2 = \dfrac{1}{9}h.$